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Spin-Zero Hawking Radiation: Bounds on the Zero-Angular-Momentum Mode Emission from Myers-Perry Black Holes

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ABSTRACT

From the quantum point of view, black holes are unstable and emit so-called Hawking radiation. Specifically, the Myers-Perry black holes are generalized rotating Kerr black holes in higher-dimensions, popular in both Kaluza-Klein and braneworld scenarios, which might in principle be detected through their Hawking radiation. One specific black hole characteristic is the greybody factor, defined in terms of the transmission probability of Hawking radiation back-scattered from the black hole gravitational potential barrier. In this paper, some rigorous bounds on the greybody factor for spin-zero Hawking radiation emitted in the zero-angular-momentum mode from the Myers-Perry black holes are calculated. This calculation serves as a template for other angular momentum modes.

Keywords: Hawking Radiation, Greybody Factors, Myers-Perry black holes, Rigorous Bounds

1. Introduction

Classically anything and everything, even light, which enters a black hole cannot escape. As a consequence, no one can (directly) see the black hole. However from the quantum point of view, black holes are unstable and emit so-called Hawking radiation, see ref. Hawking (1975). When Hawking radiation propagates in the black hole spacetime, it is modified by the curvature of spacetime resulting from that black hole. In particular, when Hawking radiation is back scattered from the black hole gravitational potential barrier, only the transmitted radiation can be observed from spatial infinity. This modified Hawking radiation, therefore, can be thought of as greybody radiation. The quantity known as the greybody factor is defined in terms of the transmission probability.

In this paper some rigorous bounds are calculated for the greybody factors for spin-zero Hawking radiation, emitted in the zero-angular-momentum mode from Myers-Perry black holes.

2. Myers-Perry Black Holes

The Myers-Perry black holes are the generalization of four-dimensional Kerr black holes to $(4+n)$ dimensions. The $(4+n)$-dimensional Myers-Perry black holes can be described by the $(4+n)$-dimensional Myers-Perry metric (Myers and Perry, 1986, Emparan and Reall, 2008)

$$ds^2 = -dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 + \frac{\mu}{r^{n-1}\Sigma}(dt - a\sin^2\theta d\phi)^2 + r^2\cos^2\theta d\Omega_n^2, \tag{1}$$

where

$$\Delta = r^2 + a^2 - \frac{\mu}{r^{n-1}}, \Sigma = r^2 + a^2\cos^2\theta, \tag{2}$$

and $d\Omega_n^2$ is the metric on n-sphere which is given by

$$d\Omega_n^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1\sin^2\theta_2 d\theta_3^2 + \ldots + \left(\prod_{i=1}^{n-1}\sin^2\theta_i\right)d\theta_n^2. \tag{3}$$

Here μ is a free parameter that determines the mass and angular momentum of the black hole. In particular, the mass and angular momentum of the black hole are defined by

$$M_{\mathrm{BH}} = \frac{(n+2)A_{n+2}}{16\pi G}\mu \quad \text{and} \quad J = \frac{2a}{n+2}M_{\mathrm{BH}}, \tag{4}$$

where A_{n+2} is the area of an $(n+2)$-dimensional unit sphere which is given by

$$A_{n+2} = \frac{2\pi^{(n+3)/2}}{\Gamma[(n+3)/2]}. \tag{5}$$

The event horizon is located at $r_{\rm H}$ which can be found from $\Delta(r_{\rm H}) = 0$. We are interested in spin zero (scalar field) Hawking radiation emitted from Myers-Perry black holes. The equation of motion for scalar fields on the Myers-Perry black hole background takes the form

$$\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi\right) = 0. \tag{6}$$

By separation of variables,

$$\Phi(t, r, \theta, \phi, \theta_1, \ldots, \theta_n) = e^{-i\omega t} e^{im\phi} \tilde{R}_{j\ell m}(r) S_{\ell m}(\theta) Y_{jn}(\theta_1, \ldots, \theta_n), \tag{7}$$

the radial equation is given by (Boonserm et al., 2014a)

$$\left[\frac{d^2}{dr_*^2} - U_{j\ell m}(r)\right] R_{j\ell m}(r) = 0. \tag{8}$$

Here r_* is the tortoise coordinate given by

$$dr_* = \frac{r^2 + a^2}{\Delta(r)} dr. \tag{9}$$

This can explicitly be expressed as

$$r_* = \int_{r_{\rm H}}^{r} \frac{r^2 + a^2}{\Delta(r)} dr \sim A_n \ln(r - r_{\rm H}) + B_n(r). \tag{10}$$

The quantity $U_{j\ell m}(r)$ is the Teukolsky potential given by

$$\begin{aligned} U_{j\ell m}(r) \quad &= \quad \frac{\Delta(r)}{(r^2+a^2)^2} \left[\lambda_{j\ell m} + \frac{j(j+n-1)a^2}{r^2} + \frac{n(n-2)\Delta(r)}{4r^2} + \frac{n\Delta'(r)}{2r} \right. \\ &\left. - \frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{(r\Delta(r))'}{r^2+a^2}\right] - \left(\omega - \frac{ma}{r^2+a^2}\right)^2. \end{aligned} \tag{11}$$

Here $\lambda_{j\ell m}$ is the separation constant. In this work, we are interested in the zero-angular-momentum mode ($m = 0$). Therefore, the Teukolsky potential becomes

$$\begin{aligned} U_{j\ell,m=0}(r) \quad &= \quad \frac{\Delta(r)}{(r^2+a^2)^2} \left[\lambda_{j\ell,m=0} + \frac{j(j+n-1)a^2}{r^2} + \frac{n(n-2)\Delta(r)}{4r^2} \right. \\ &\left. + \frac{n\Delta'(r)}{2r} - \frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{(r\Delta(r))'}{r^2+a^2}\right] - \omega^2. \end{aligned} \tag{12}$$

We can rewrite the Teukolsky potential as

$$U_{j\ell,m=0}(r) = V_{j\ell,m=0}(r) - \omega^2, \tag{13}$$

where

$$\begin{aligned} V_{j\ell,m=0}(r) &= \frac{\Delta(r)}{(r^2+a^2)^2}\left[\lambda_{j\ell,m=0} + \frac{j(j+n-1)a^2}{r^2} + \frac{n(n-2)\Delta(r)}{4r^2}\right. \\ &\left. + \frac{n\Delta'(r)}{2r} - \frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{(r\Delta(r))'}{r^2+a^2}\right]. \end{aligned} \tag{14}$$

Figures 1 and 2 shows the potential $V_{j\ell,m=0}(r)$ in five $(n=1)$ and six $(n=2)$ dimensions.

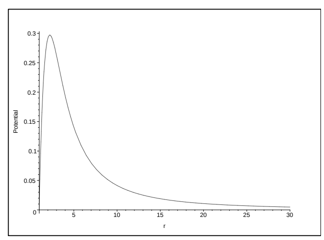

Figure 1: The Myers-Perry potential for $n = 1$.

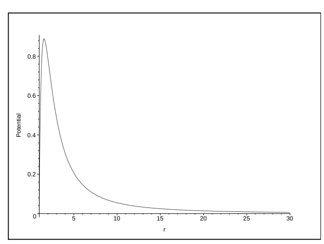

Figure 2: The Myers-Perry potential for $n = 2$.

3. Rigorous Bounds on Greybody Factors

In general, the exact greybody factors are impossible to obtain even for the Schwarzschild black hole, which is by far the simplest case. Thus, it is of interest to develop new methods in calculating the greybody factors. One of them is to place some rigorous bounds on the greybody factors. The relevant bounds were first developed in Visser (1999). They were further developed in

Boonserm and Visser (2008a, 2009), Boonserm (2009), Boonserm and Visser (2010a,b) These bounds have been specifically applied to black hole systems (Boonserm and Visser, 2008b, Ngampitipan and Boonserm, 2013a,b, Boonserm et al., 2013, 2014b). General and robust bounds on the greybody factors are given by (Visser, 1999, Boonserm and Visser, 2008a, 2009)

$$T_{j\ell m} \geq \operatorname{sech}^2\left(\int_{-\infty}^{\infty} \vartheta dr_*\right), \tag{15}$$

where

$$\vartheta = \frac{\sqrt{[h'(r_*)]^2 + [U_{j\ell m}(r_*) + h^2(r_*)]^2}}{2h(r_*)}. \tag{16}$$

and $h(r_*)$ is any positive function. We choose $h(r_*) = \omega$ and consider the $m = 0$ case. Then,

$$\begin{aligned} T \quad \geq \quad & \operatorname{sech}^2\left[\frac{1}{2\omega}\int_{r_{\mathrm{H}}}^{\infty}\left|\frac{1}{r^2+a^2}\left\{\lambda_{j\ell,m=0} + \frac{j(j+n-1)a^2}{r^2} + \frac{n(n-2)\Delta(r)}{4r^2}\right.\right.\right. \\ & \left.\left.\left. + \frac{n\Delta'(r)}{2r} - \frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{(r\Delta(r))'}{r^2+a^2}\right\}\right| dr\right]. \end{aligned} \tag{17}$$

We can show that the argument of the absolute value is positive for $r > r_{\mathrm{H}}$. Thus, we can write

$$\begin{aligned} T \quad \geq \quad & \operatorname{sech}^2\left[\frac{1}{2\omega}\int_{r_{\mathrm{H}}}^{\infty}\frac{1}{r^2+a^2}\left\{\lambda_{j\ell,m=0} + \frac{j(j+n-1)a^2}{r^2} + \frac{n(n-2)\Delta(r)}{4r^2}\right.\right. \\ & \left.\left. + \frac{n\Delta'(r)}{2r} - \frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{(r\Delta(r))'}{r^2+a^2}\right\} dr\right]. \end{aligned} \tag{18}$$

Performing the first integral, we obtain

$$\int_{r_{\mathrm{H}}}^{\infty} \frac{\lambda_{j\ell,m=0}}{r^2+a^2} dr = \frac{\lambda_{j\ell,m=0}}{a} \arctan \frac{r}{a}\bigg|_{r_{\mathrm{H}}}^{\infty} = \frac{\lambda_{j\ell,m=0}}{a} \arctan \frac{a}{r_{\mathrm{H}}}. \tag{19}$$

By integrating by parts, we can show that

$$\int_{r_{\mathrm{H}}}^{\infty} \frac{1}{r^2+a^2}\left[-\frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{(r\Delta(r))'}{r^2+a^2}\right] dr = \int_{r_{\mathrm{H}}}^{\infty} \frac{r^2\Delta(r)}{(r^2+a^2)^3} dr. \tag{20}$$

This integral can be explicitly performed and gives

$$\begin{aligned} \int_{r_{\mathrm{H}}}^{\infty} \frac{r^2\Delta(r)}{(r^2+a^2)^3} dr \quad = \quad & \frac{n}{8r_{\mathrm{H}}} - \frac{n(n-2)\left(r_{\mathrm{H}}^2+a^2\right)}{8(n+2)r_{\mathrm{H}}^3}\,{}_2F_1\left(1, \frac{n+2}{2}, \frac{n+4}{2}, -\frac{a^2}{r_{\mathrm{H}}^2}\right) \\ & - \frac{a^2}{4r_{\mathrm{H}}\left(r_{\mathrm{H}}^2+a^2\right)} + \frac{1}{2a}\arctan\frac{a}{r_{\mathrm{H}}}. \end{aligned} \tag{21}$$

Here ${}_2F_1\left(z_1, z_2, z_3, z_4\right)$ is the hypergeometric function. The j-dependent integral yields

$$\int_{r_{\mathrm{H}}}^{\infty} \frac{j(j+n-1)a^2}{r^2\left(r^2+a^2\right)} dr = \frac{j(j+n-1)}{r_{\mathrm{H}}} - \frac{j(j+n-1)}{a} \arctan \frac{a}{r_{\mathrm{H}}}. \tag{22}$$

Calculating the n-dependent integral, we obtain

$$\begin{aligned} \int_{r_{\mathrm{H}}}^{\infty} \frac{1}{r^2+a^2}\left[\frac{n(n-2)\Delta(r)}{4r^2} + \frac{n\Delta'(r)}{2r}\right] dr \quad = \quad & \frac{n^2\left(r_{\mathrm{H}}^2+a^2\right)}{4(n+2)r_{\mathrm{H}}^3} {}_2F_1\left(1, \frac{n+2}{2}, \frac{n+4}{2}, -\frac{a^2}{r_{\mathrm{H}}^2}\right) \\ & + \frac{n(n-2)}{4r_{\mathrm{H}}} + \frac{n}{a} \arctan \frac{a}{r_{\mathrm{H}}}. \end{aligned} \tag{23}$$

Collecting all the results, we obtain

$$T_{j\ell,m=0} \geq \operatorname{sech}^2 \left| \frac{1}{2\omega r_{\mathrm{H}}} I_{j\ell,m=0} \right|. \tag{24}$$

Here

$$\begin{aligned} I_{j\ell,m=0} \quad = \quad & \frac{n(2n-3)}{8} + j(j+n-1) + \frac{n\left(r_{\mathrm{H}}^2+a^2\right)}{8r_{\mathrm{H}}^2} {}_2F_1\left(1, \frac{n+2}{2}, \frac{n+4}{2}, -\frac{a^2}{r_{\mathrm{H}}^2}\right) \\ & + \frac{a^2}{4\left(r_{\mathrm{H}}^2+a^2\right)} + \left[\frac{2n+1}{2} - j(j+n-1) + \lambda_{j\ell,m=0}\right] \frac{r_{\mathrm{H}}}{a} \arctan \frac{a}{r_{\mathrm{H}}} \end{aligned} \tag{25}$$

In the limit $a \to 0$, $n = 0$ and $j = 0$, we obtain

$$\lim_{a\to 0} I_{j=0,\ell,m=0} = \lim_{a\to 0}\left[-\frac{a^2}{4\left(r_{\mathrm{H}}^2+a^2\right)} + \left(\frac{1}{2} + \lambda_{j=0,\ell,m=0}\right) \frac{r_{\mathrm{H}}}{a} \arctan \frac{a}{r_{\mathrm{H}}}\right] = \frac{1}{2} + \lambda_{j=0,\ell,m=0}. \tag{26}$$

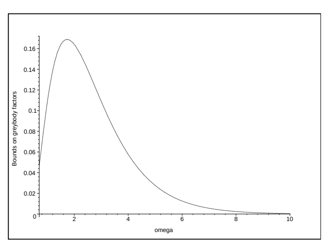

Figure 3: The bounds on the greybody factors as a function of ω for $n = 1$.

Figures 3 and 4 show the bounds on the greybody factors as a function of ω in five ($n = 1$) and six ($n = 2$) dimensions, respectively. Figures 5 and 6 show the bounds on the greybody factors as a function of the black hole angular momentum in five ($n = 1$) and six ($n = 2$) dimensions, respectively.

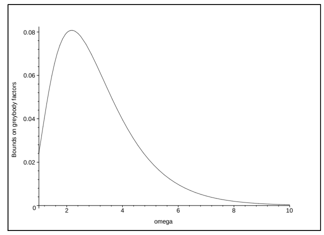

Figure 4: The bounds on the greybody factors as a function of ω for $n = 2$.

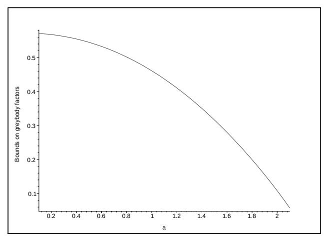

Figure 5: The bounds on the greybody factors as a function of the black hole angular momentum for $n = 1$.

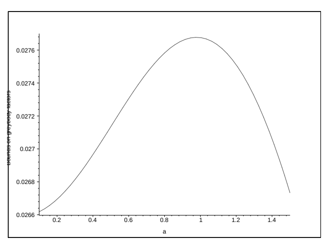

Figure 6: The bounds on the greybody factors as a function of the black hole angular momentum for $n = 2$.

4. Conclusion

In this paper, we have obtained rigorous bounds on the greybody factors for spin-zero Hawking radiation emitted in the zero-angular-momentum mode from the Myers-Perry black holes. Qualitatively, the bounds seem to decrease in higher dimensions. In five dimensions corresponding to n = 1, the bounds decrease when increasing the black hole angular momentum. In six dimensions corresponding to n = 2, the bounds increase to reach the maximum and start to decrease when increasing the black hole angular momentum.

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